mined immediately for two cases:

Case I:

$$\text{If } L'(p_1) \le 0 \qquad \text{then } x^* = p_1 \tag{11}$$

Case II:

$$\text{If } L'(p_2) \ge 0 \qquad \text{then } x^* = p_1 \tag{12}$$

Because $L'' < 0$, case I corresponds to a constrained global maximum of $L$ at $x = p_1$ and the desired constrained global minimum at $x = p_2$. The necessary conditions of Eq. (9) are satisfied by $\lambda_1 = 0$ and $\lambda_2 = -L'(p_1) \ge 0$. Case II represents a constrained global maximum at $x = p_2$ and the desired constrained global minimum at $x = p_1$. The necessary conditions are satisfied by $\lambda_2 = 0$ and $\lambda_1 = L'(p_2) \ge 0$.

The condition of Eq. (11) for case I can be stated explicitly using the expression for $L'$ in Eq. (10) as

$$K \le C_2 \ell n\alpha + \left(1 + \frac{c_2}{c_1}\right)p_1 \tag{13}$$

where $\alpha = (m_2/m_1)/(c_2/c_1)$. Note that when $p_1 = 0$, i.e., $K$ is sufficiently small that the propellant mass fraction constraint of vehicle 1 is inactive, Eq. (13) can be satisfied only if $\alpha > 1$. The condition of Eq. (12) for case II requires that

$$K \ge c_2 \ell n\alpha + \left(1 + \frac{c_2}{c_1}\right)p_2 \tag{14}$$

Note that when $p_2 = K$, i.e., $K$ is sufficiently small that the propellant mass fraction constraint of vehicle 2 is inactive, Eq. (14) can be satisfied only if $\alpha < 1$.

Besides cases I and II, the only other possible situation is Case III, for which $L'(p_1) > 0$ and $L'(p_2) < 0$. In this case one solution to the necessary conditions of Eq. (9) is $\lambda_1 = \lambda_2 = 0$ (both constraints inactive) and $L'(x^*) = 0$. However, because $L'' < 0$, this stationary value is a global maximum solution which must be discarded. The desired solution is one of the two constrained minima which exist at $x = p_1$ and $p_2$, for which the respective solutions to the necessary conditions are $\lambda_2 = 0$ along with $\lambda_1 = L'(p_1) \ge 0$ and $\lambda_1 = 0$ along with $\lambda_2 = -L'(p_2) \ge 0$. Whichever local minimum provides the smaller value of $L$ is the global constrained minimum.

## III. Numerical Examples

The four numerical examples that follow illustrate the variety and complexity of the optimal solutions. In all of the examples, $m_2/m_1 = 1.5$ and $c_2/c_1 = 1.25$, resulting in $\alpha = 1.2$. In the first example, $K/c_1 = 0.2$ with the propellant mass fractions given by $\beta_1 = 0.3$ and $\beta_2 = 0.2$. Because the required velocity change $K$ is sufficiently small, $p_1 = 0$ and $p_2 = K$ in Eq. (7) and the condition of Eq. (11) representing case I is satisfied. Thus, $x^* = K$ is the constrained optimal solution, indicating that $\Delta V_1 = K$ and $\Delta V_2 = 0$. All of the velocity change is made by vehicle 1, which has both the smaller mass and a smaller exhaust velocity.

As a second example, consider the large velocity change $K/c_1 = 1$ along with the mass fraction constraints $\beta_1 = 0.65$ and $\beta_2 = 0.52$. In this case, the mass fraction constraint of vehicle 2 is active and $p_1 = K - a_2 = 0.0825c_1$ in Eq. (7). The value of $p_2$ is $K$, because the mass fraction constraint of vehicle 1 is inactive. Because $L'(p_1) > 0$ and $L'(p_2) < 0$, this represents case III, and a direct comparison of the cost yields $x^* = p_2 = K$, indicating $\Delta V_1 = K$ and $\Delta V_2 = 0$.

In the third example, $K/c_1 = 0.8$ and $\beta_1 = \beta_2 = 0.5$. In this case, $p_1 = 0$ and $p_2 = a_1 = 0.693c_1$ because only the vehicle 2 mass fraction constraint is active. This is an example of case III for which the optimal solution is $x^* = 0.693c_1$, indicating that both vehicles provide velocity change: $\Delta V_1 = 0.693c_1$, and $\Delta V_2 = 0.107c_1$.

The fourth and final example is the same as the second example except that $\beta_1 = 0.3$ rather than 0.65. This has the effect of making both propellant mass fraction constraints active. This effect automatically requires both vehicles to provide velocity change because neither has enough propellant to perform the total maneuver. When the costs of the two constrained local minima are compared for this case III solution, the optimal solution is $x^* = p_1 = 0.0825c_1$. Thus, $\Delta V_1 = 0.0825c_1$ and $\Delta V_2 = 0.9175c_1$, which is a surprising result compared to the second example. In that example, vehicle 2 did not have enough propellant to perform the rendezvous and vehicle 1 provided the total velocity change. When neither vehicle has enough propellant to perform the rendezvous, the optimal solution requires that vehicle 2 provides over 90% of the velocity change.

The explanation of this apparent paradox lies in the fact that in the fourth example, vehicle 2 was forced to provide part of the velocity change because vehicle 1 did not have enough propellant. The mass decrease of vehicle 2 incurred in making up the velocity change deficit of vehicle 1 is enough to make vehicle 2 the more propellant-efficient vehicle of the two. It is therefore optimal to have vehicle 2 make up the entire deficit velocity change of vehicle 1.

## IV. Concluding Remarks

The apparently simple problem of optimal cooperative rendezvous of two active space vehicles yields solutions which are interesting due to their complexity. If neither propellant mass fraction constraint is active, one vehicle provides all of the required velocity change. If both mass fraction constraints are active, there are two possibilities: both vehicles provide velocity change, or, if the required velocity change is too large, no solution exists. In the case of one active propellant mass constraint, the optimal solution can require either one or both vehicles to provide velocity change.

## Acknowledgment

This research was supported by NASA Research Grant NAG 3-805 sponsored by the NASA Lewis Research Center.

## Reference

[1]Bryson, A. E., Jr. and Ho, Y-C, *Applied Optimal Control*, Hemisphere, Washington, DC, 1975, Chap. 1.

# Sequential Design of Discrete Linear Quadratic Regulators via Optimal Root-Locus Techniques

Sekar Ganesan* and Leang S. Shieh†
*University of Houston, Houston, Texas*
and
Robert E. Yates‡
*U.S. Army Missile Command, Redstone, Alabama*

## I. Introduction

THE asymptotic behavior of the optimal root-loci (frequency-domain approach) of linear time-invariant continuous-time control systems has been discussed by Chang[1] and Kalman[2] for the single-input case, and the multi-

Received Oct. 13, 1987; revision received Feb. 8, 1988. 

*Graduate Student, Department of Electrical Engineering.
†Professor, Department of Electrical Engineering.
‡Director of Guidance and Control Directorate.

input case has been dealt with by Tyler and Tuteur,[3] Kwakernaak and Sivan,[4] Kwakernaak,[5] Kouvaritakis,[6] Harvey and Stein,[7] and Grimble.[8] Many useful results regarding the asymptotic behavior of the poles and zeros of the open-loop system and those of the optimally regulated closed-loop system have been obtained. However, very few methods deal with the inverse optimal control problem. Pal and Mahalanabis[9] proposed an optimal feedback controller with specified relative stability for single-input systems based on the frequency-domain optimality condition. Recently, Harvey and Stein[7] have developed procedures for selecting the weighting matrices based on the asymptotic properties of the desired state regulator, whereas Grimble[8] has considered the design of optimal output regulators using multivariable root-loci in the complex $s$ plane.

The discrete-time counterpart of this optimal control problem, based on the asymptotic properties introduced by Letov[10] and Chang,[1] has also been dealt with by Kwakernaak[5] and Lewis.[11] But again, unlike the time-domain optimal pole placement problem, the frequency-domain discrete linear quadratic regulator problem with desirable pole placement in the complex $z$ plane has received very little attention.

In this Note, we present a sequential design method for determining the quadratic weighting matrices and state feedback regulators for discrete-time systems with the help of optimal root-locus techniques. The closed-loop system is designed so that it may retain some stable open-loop poles and the associated open-loop eigenvectors. The remaining closed-loop poles can be optimally placed at desired locations and/or specified regions of the complex $z$ plane.

## II. Optimal State Feedback Gain and Closed-Loop Eigensystem

Consider the stabilizable and detectable linear time-invariant discrete system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \tag{1}$$

and the quadratic performance index

$$J = \frac{1}{2}\left[\sum_{i=0}^{\infty} x^T(i)Qx(i) + u^T(i)Ru(i)\right] \tag{2}$$

where $\boldsymbol{x}(k)$ and $\boldsymbol{u}(k)$ are $n \times 1$ state vector and $m \times 1$ input vector, respectively, and $A$ and $B$ are constant matrices of appropriate dimensions. Also, the weighting matrices $Q$ and $R$ are nonnegative definite and positive definite symmetric matrices, respectively.

Let the eigenspectrum of the open-loop system matrix of $A$ be $\sigma(A) = \{\lambda_i, i = 1,...,n\}$. It is well known[4,11] that the performance index in Eq. (2) is minimized if the feedback control law is chosen as

$$\boldsymbol{u}(k) = -(B^TPB + R)^{-1}B^TPA\boldsymbol{x}(k) + r(k) \triangleq -K\boldsymbol{x}(k) + r(k) \tag{3}$$

where $r(k)$ is an $m \times 1$ reference input and the optimal state feedback gain is $K$. Also, $P$ is the unique nonnegative definite solution of the discrete Riccati equation

$$P = A^T[P - PB(B^TPB + R)^{-1}B^TP]A + Q \tag{4}$$

The resulting closed-loop system becomes

$$\boldsymbol{x}(k+1) = A_c\boldsymbol{x}(k) + Br(k) \tag{5}$$

where the closed-loop system matrix $A_c = A - BK$. Since it is assumed that the open-loop system in Eq. (1) satisfies the stabilizability and detectability conditions, the closed-loop system in Eq. (5) is asymptotically stable, i.e., the eigenvalues of $A_c$ given by $\sigma(A_c) = \{\alpha_i, i = 1,...,n\}$ are located inside the unit circle. The design method presented in this Note is based on the Chang-Letov equation[11]

$$\begin{aligned}\Delta_c(z^{-1})\Delta_c(z) &= |H^T(z^{-1})QH(z) + R|\Delta(z^{-1})\Delta(z) \\ &\times |B^TPB + R|^{-1}\end{aligned} \tag{6}$$

where $\Delta_c(z)$ and $\Delta(z)(=|zI_n - A|)$ are the closed-loop and open-loop characteristic polynomials, respectively, and $|(.)|$ represents the determinant of (.). $H(z)$ in Eq. (6) is the open-loop transfer function matrix of the autonomous part of the system in Eq. (1), i.e., $H(z) = (zI_n - A)^{-1}B \triangleq N(z)/\Delta(z)$, where $N(z) = \text{adj}(zI_n - A)B$. Note that $H(z)$, $N(z)$ and the open-loop characteristic polynomial $\Delta(z)$ can be computed[12] given the plant in Eq. (1). In the following, some relations between the optimal state feedback gain $K$, the optimal closed-loop eigenvalues $\alpha_i$, and the corresponding closed-loop eigenvectors $\xi_i$ and the weighting matrix $Q$ are described.

*Lemma 1*[13]: Consider the open-loop system in Eq. (1) with the eigenvalues $\lambda_i, i = 1,...,n$, and performance index in Eq. (2). Also, let the optimal closed-loop system in Eq. (5) have eigenvalues $\alpha_i, i = 1,...,n$, with $\alpha_i \neq \alpha_j\ (i \neq j)$, $\alpha_i \neq \lambda_i$, and the corresponding eigenvectors $\xi_i$. Then, the optimal state feedback gain $K$ is given by

$$K = -[\rho_1,...,\rho_n][\xi_1,...,\xi_n]^{-1} \tag{7a}$$

where

$$\rho_i = \Delta(\alpha_i)w_i \quad \text{and} \quad \xi_i = N(\alpha_i)w_i \tag{7b}$$

and $w_i$ is any nonzero vector satisfying

$$[\Delta(\alpha_i^{-1})R\Delta(\alpha_i) + N^T(\alpha_i^{-1})QN(\alpha_i)]w_i = 0_{m\times 1} \tag{7c}$$

*Lemma 2*: Let the model matrix $M$ of the system matrix $A$ in Eq. (1) be decomposed into two block eigenvectors, i.e., $M = [M_x, M_y]$, with $M_x \triangleq [M_1^-,...,M_{n^-}^-]$ and $M_y \triangleq [M_1^+,..., M_{n^+}^+]$. The column vectors $M_i^-$ $(i = 1,...,n^-)$ and $M_i^+$ $(i = 1,...n^+)$ are the eigenvectors associated with the eigenvalues $\lambda_i^-$ $(i = 1,...,n^-)$ and $\lambda_i^+$ $(i = 1,...,n^+)$, respectively, with $n = n^- + n^+$. The open-loop eigenvalues could be simple or repeated and the corresponding eigenvectors would be determined appropriately. Also, let $\lambda_i^-$ and $M_i^-$ be the stable open-loop eigenvalues and the associated eigenvectors, respectively, that need to be kept invariant in the closed-loop system [Eq. (5)], i.e., $\alpha_i = \lambda_i^-$ and $\xi_i = M_i^-$ for $i = 1,...,n^-$, where $\alpha_i$ and $\xi_i$ are the eigenvalues and the associated eigenvectors of the closed-loop system. Now, if a $Q$ matrix is constructed such that

$$QM_i^- = 0_{n\times 1}, i = 1,...,n^- \quad \text{or} \quad \text{null}(Q) = \text{span}(M_1^-,...,M_{n^-}^-) \tag{8a}$$

then the optimal state feedback gain $K$ in Eq. (7a) becomes

$$\begin{aligned}K = &-[0_{m\times 1},...,0_{m\times 1},\rho_{n^-+1},...\rho_n] \\ &\times [M_1^-,...,M_{n^-}^-,\xi_{n^-+1},...,\xi_n]^{-1}\end{aligned} \tag{8b}$$

and the optimal closed-loop system in Eq. (5) contains the invariant eigenvalues $\lambda_i^-$ and their associated eigenvectors $M_i^-$ for $i = 1,...,n^-$.

*Proof*: For $\lambda_i \neq \lambda_j\ (i \neq j)$, we can write

$$(\lambda_i^- I_n - A)\ \text{adj}(\lambda_i^- I_n - A) = \Delta(\lambda_i^-)I_n = 0_n \tag{9a}$$

Thus, the eigenvector $M_i^-$ of $A$ associated with $\lambda_i^-$ is a nonzero column vector or a linear combination of the column vectors of $\text{adj}(\lambda_i^- I_n - A)$ for $\lambda_i^- \neq \lambda_j^-$. Also, from Eqs. (7b) and (9a), we have

$$\xi_i = N(\lambda_i^-)w_i = \text{adj}(\lambda_i^- I_n - A)Bw_i = M_i^- \quad \text{for} \quad i = 1,...,n^- \tag{9b}$$

Furthermore, from Eqs. (7c) and (9b), we obtain

$$\left[\Delta\left(\frac{1}{\lambda_i^-}\right)R\Delta(\lambda_i^-)+N^T\left(\frac{1}{\lambda_i^-}\right)QN(\lambda_i^-)\right]w_i=0_{m\times 1} \quad (9c)$$

or

$$N^T\left(\frac{1}{\lambda_i^-}\right)QN(\lambda_i^-)w_i=N^T\left(\frac{1}{\lambda_i^-}\right)Q\xi_i=N^T\left(\frac{1}{\lambda_i^-}\right)QM_i^-=0_{m\times 1} \quad (9d)$$

Hence, if we choose $Q$ such that $Q\xi_i=QM_i^-=0_{n\times 1}$, then $w_i$ can be any nonzero vector. Since, $\rho_i=\Delta(\alpha_i)w_i=\Delta(\lambda_i^-)w_i=0_{m\times 1}$, from Eq. (7b), the optimal feedback gain $K$ can be written as given in Eq. (8b). Thus, the closed-loop system in Eq. (5) obtained using the state feedback gain $K$ in Eq. (8b) retains the eigenvalues $\lambda_i^-$ and the associated eigenvectors $M_i^-$ for $i=1,...,n^-$. When $\lambda_i^-=\lambda_j^-$, the above result can be proved in a similar manner.

*Lemma 3*: Let $M$ be the modal matrix of $A$ defined in Lemma 2 and let the inverse of $M$ be given by $M^{-1}=[M_x,M_y]^{-1}\triangleq[\hat{M}_x^T,\hat{M}_y^T]^T$, where $M_x\in R^{n\times n^-}$, $M_y\in R^{n\times n^+}$, $\hat{M}_x\in R^{n^-\times n}$, and $\hat{M}_y\in R^{n^+\times n}$. Then the desired $Q$ matrix that satisfied Eq. (8a) becomes

$$Q=q\hat{M}_y^TD\hat{M}_y \quad (10)$$

where $q$ is a positive scalar and $D$ is an $n^+\times n^+$ nonnegative definite symmetric matrix.

*Proof*: Since $M^{-1}M=I_n$, it can be easily seen that $\hat{M}_yM_x=0_{n^+\times n^-}$. Thus, we have $QM_x=q\hat{M}_y^TD\hat{M}_yM_x=0_{n\times n^-}$.

## III. Discrete LQR in the Frequency Domain

*Theorem*: Consider the discrete multivariable system given in Eq. (1) [its transfer function matrix $H(z)$] and the associated performance index in Eq. (2). Let $\bar{\lambda}_i$ and $M_i^-$ $(i=1,...,n^-)$ be the stable open-loop eigenvalues and the eigenvectors that are to be retained in the closed-loop system in Eq. (5). The remaining eigenvalues and eigenvectors of the closed-loop system are defined as $\lambda_i^+$ and $M_i^+$ $(i=1,...,n^+,\ n^+=n-n^-)$, respectively. Let the modal matrix $M$ of $A$ be decomposed into block eigenvectors as defined in Lemma 2 and let $M^{-1}$ be composed of block vectors $\hat{M}_x$ and $\hat{M}_y$. Also, let the weighting matrices for the performance index in Eq. (2) be $R=r\hat{R}$ $(\hat{R}>0)$ and $Q$, as in Eq. (10), with $D=\boldsymbol{dd}^T$, where $\boldsymbol{d}\in R^{n^+\times 1}$ is an arbitrary nonzero column vector and $r$ and $q$ are two positive scalars to be determined. The optimal closed-loop eigenvalues can be determined from the root-loci of the following equation, with the scalar gain $q/r$ as a variable

$$\Delta_c(z^{-1})\Delta_c(z)=[\Delta(z^{-1})\Delta(z)+(q/r)\beta(z)\beta^T(z^{-1})]W \quad (11a)$$

where

$$\beta(z)\beta^T(z^{-1})=[\boldsymbol{d}^T\hat{M}_yN(z)\hat{R}^{-1/2}][\boldsymbol{d}^T\hat{M}_yN(z^{-1})\hat{R}^{-1/2}]^T \quad (11b)$$

and $W=|I_m+\hat{R}^{-1}B^TPB|^{-1}$. Note that though $W$ involves $P$, the solution of the discrete Riccati equation in Eq. (4) is a constant and can be neglected since it does not affect the determination of the roots of the closed-loop characteristic polynomial $\Delta_c(z)$. The vector $\boldsymbol{d}$ can be chosen to fix some virtual open-loop zeros that are the finite asymptotic poles of a virtual closed-loop system as $q/r\to\infty$, so that the optimal closed-loop poles can be placed in desired positions in the complex $z$ plane.

*Proof*: From Eq. (6), we have

$$\begin{aligned}\Delta_c(z^{-1})\Delta_c(z)&=\Delta(z^{-1})\Delta(z)|[I_m+(q/r)\hat{R}^{-1/2}H^T(z^{-1})\\&\quad\times\hat{M}_y^T\boldsymbol{dd}^T\hat{M}_yH(z)\hat{R}^{-1/2}]|\\&=\Delta(z^{-1})\Delta(z)+(q/r)[\boldsymbol{d}^T\hat{M}_yN(z)\hat{R}^{-1/2}]\\&\quad\times[\boldsymbol{d}^T\hat{M}_yN(z^{-1})\hat{R}^{-1/2}]^T\\&=\Delta(z^{-1})\Delta(z)+(q/r)\beta(z)\beta^T(z^{-1})\end{aligned} \quad (12)$$

When $(q/r)\to\infty$, the stable zeros of $\beta(z)\beta^T(z^{-1})$ are the zeros of $\Delta_c(z)$, i.e., the finite asymptotic poles of the closed-loop system. Hence, by assigning the vector $\boldsymbol{d}$ appropriately, the stable zeros of $\beta(z)\beta^T(z^{-1})$ can be placed at some desired locations in the complex $z$ plane. Then, $q$ and $r$ can be chosen so that the final closed-loop poles of the system are in desirable regions. Sometimes, it might be possible to assign the closed-loop poles exactly.

*Remark 1*: For sequential design, the weighting matrix $R$ has to be updated as $R_{i+1}=R_i+B^TP_iB$[14] at each step, so as to render the final closed-loop system optimal with respect to $R_1$ and the summation of all $Q_i$.

*Sequential Design Procedure*

*Step 1*: Let the given discrete multivariable system be as in Eq. (1), with $H(z)$ being its transfer function matrix. Set an index $j=1$. Specify $R$ and let $\hat{R}=R$, i.e., $r=1$. Also, set $\hat{Q}=0_n$ and $\hat{K}=0_{m\times n}$.

*Step 2*: Let $\lambda_i^-, i=1,...,n^-$ be the eigenvalues to be kept invariant at this stage and also let the variant eigenvalues be $\lambda_i^+, i=1,...,n^+$, with $n=n^-+n^+$. Find the block vector $\hat{M}_y\in R^{n^+\times n}$, from the inverse of the modal matrix $M$ (of $A$).

*Step 3*: Assign the $n^+\times 1$ column vector $\boldsymbol{d}$ to determine the desired zeros of Eq. (11b). Draw the root-loci of Eq. (11a) with $q$ as the variable and then select a suitable $q$ so that a desired set of closed-loop poles $\alpha_i$ can be obtained. Compute the weighting matrix $Q=q\hat{M}_y^T\boldsymbol{dd}^T\hat{M}_y$ and then determine the feedback gain $K$ from Eq. (8b) or from the solution of the discrete Riccati equation in Eq. (4).

*Step 4*: Update $H(z):=H(z)[I_m+KH(z)]^{-1}$, $A:=A-BK$, $R:=R+B^TPB$, $\hat{Q}:=\hat{Q}+Q$, and $\hat{K}:=\hat{K}+K$. Set $j:=j+1$. Now, if all the optimal closed-loop eigenvalues have been determined, or if $j=L\le n$, go to step 5; else, go to step 2.

*Step 5*: The final closed-loop system is $A$ with its total optimal feedback gain being $\hat{K}$. This gain is optimal with respect to the performance index in Eq. (2) for the weighting matrices $Q=\hat{Q}$ and $R=\hat{R}$.

## IV. Illustrative Example

Consider the discrete system in Eq. (1) with

$$A=\begin{bmatrix}-0.2&0&1.4\\-0.6&0.4&1.4\\-0.8&0.1&1.2\end{bmatrix},\qquad B=\begin{bmatrix}0&0\\-1&0\\1&-1\end{bmatrix} \quad (13)$$

Then

$$H(z)=N(z)/\Delta(z)$$

$$=\frac{1}{\Delta(z)}\begin{bmatrix}1.4(z-0.5)&-1.4(z-0.4)\\-z^2+2.4z-1.44&-1.4(z-0.4)\\z^2-0.3z-0.1&-z^2+0.2z+0.08\end{bmatrix} \quad (14)$$

and $\Delta(z)=(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)$, with $\lambda_1=0.4$ and $\lambda_{2,3}=0.5\pm j0.7$. The objective is to keep the eigenvalue at 0.4 invariant and move the complex conjugate pair to desirable locations on the positive real axis. This will result in a stable closed-loop system devoid of any oscillating modes.

Following the design procedure, we let $j = 1$. The input weighting matrix is specified as $R = \hat{R} = \text{diag}\,[3,1]$. According to our notation, $(\lambda_i^-) = 0.4$ and $(\lambda_{1,2}^+) = 0.5 \pm j0.7$. From the inverse of the block modal matrix $M$, we obtain

$$\hat{M}_y = \begin{bmatrix} 0.1712 & -0.0112 & -0.28 \\ 0.1584 & -0.0384 & 0.04 \end{bmatrix} \tag{15}$$

To relocate the poles $\lambda_{2,3}$ on the positive real axis, we choose the column vector $\boldsymbol{d}$ as $\boldsymbol{d} = [1,0]^T$, so that the asymptotic poles of the closed-loop system tend to 0 and 0.6225 as $q \to \infty$, i.e.,

$$\beta(z)\beta^T(z^{-1}) = 0.103(z - 0.4)(z^{-1} - 0.4) \times (z - 0.6225)(z^{-1} - 0.6225)$$

Selecting $q = 60$, the variant closed-loop poles become $\alpha_2 = 0.215$ and $\alpha_3 = 0.45$. The state weighting matrix $Q$ is

$$Q = q\hat{M}_y^T \boldsymbol{d}\boldsymbol{d}^T \hat{M}_y = \begin{bmatrix} 1.76 & -0.115 & -2.88 \\ -0.115 & 0.0075 & 0.19 \\ -2.88 & 0.19 & 4.7 \end{bmatrix} \tag{16}$$

For the closed-loop eigenvalue $\alpha_2 = 0.215$, the vectors $\rho_2$ and $\xi_2$ are $\rho_2 = [-0.312,1]^T$ and $\xi_2 = [-3.627,-5.312,-1.075]^T$. Similarly, for $\alpha_3 = 0.45$, we have $\rho_3 = [-0.3,1]^T$ and $\xi_3 = [-2.0,4.0,-0.925]^T$. Thus, from Eq. (8b), the state feedback gain $K$ becomes

$$K = -[0_{2\times 1},\rho_2,\rho_3][M_1^-,\xi_2,\xi_3]^{-1} = \begin{bmatrix} -0.15 & 0.023 & 0.1 \\ 0.466 & -0.077 & -0.26 \end{bmatrix} \tag{17}$$

where $M_1^- = [1.4,6.4,0.6]^T$. Note that this feedback gain can also be obtained by solving the discrete Riccati equation in Eq. (4), with $(A,B)$ and $(Q,R)$. The optimal closed-loop system is

$$A := A - BK = \begin{bmatrix} -0.2 & 0 & 1.4 \\ -0.75 & 0.423 & 1.5 \\ -0.184 & 0 & 0.84 \end{bmatrix} \tag{18}$$

with its eigenvalues being $\sigma(A_c) = \{0.215,0.4,0.45\}$. The final closed-loop system has positive real eigenvalues at desired locations and it is optimal with respect to the performance index in Eq. (2) for $(Q,R)$.

## V. Conclusion

A sequential method that uses the classical root-locus techniques has been developed to determine the quadratic weighting matrices and the discrete linear quadratic regulators of multivariable control systems. In this proposed approach, at each recursive step, an intermediate unity rank state-weighting matrix containing some invariant eigenvectors of that open-loop system matrix is assigned. Also, at each step, an intermediate characteristic equation of the closed-loop system containing the invariant eigenvalues is created. In order to control the movement of the root-loci and choose desirable closed-loop poles, some virtual finite open-loop zeros are assigned to this characteristic equation. The designed optimal closed-loop system thus would retain some stable open-loop poles and have the remaining poles optimally placed at desired locations in the complex $z$ plane.

## Acknowledgments

This work was supported in part by the U.S. Army Research Office under Contract DAAL-03-87-K-0001, the U.S. Army Missile Research and Development Command under Contract DAAH-01-85-C-A111, and the NASA Johnson Space Center under Contract NAG 9-211. The authors wish to express their gratitude for the valuable remarks and suggestions made by the reviewers.

## References

[1]Chang, S. S. L., *Synthesis of Optimal Control Systems*, McGraw-Hill, New York, 1961.
[2]Kalman, R. E., "When Is a Linear Control System Optimal?" *ASME Journal of Basic Engineering*, Vol. 86, March 1964, pp. 51–60.
[3]Tyler, J. S. and Tuteur, F. B., "The Use of a Quadratic Performance Index to Design Multivariable Control Systems," *IEEE Transactions on Automatic Control*, AC-11, Jan. 1964, pp. 84–92.
[4]Kwakernaak, H. and Sivan, R., *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.
[5]Kwakernaak, H., "Asymptotic Root-loci of Multivariable Linear Optimal Regulators," *IEEE Transactions on Automatic Control*, AC-21, June 1976, pp. 378–382.
[6]Kouvaritakis, B., "The Optimal Root-loci of Linear Multivariable Systems," *International Journal of Control*, Vol. 28, July 1978, pp. 33–62.
[7]Harvey, C. A. and Stein, G., "Quadratic Weights for Asymptotic Regulator Properties," *IEEE Transactions on Automatic Control*, AC-23, June 1978, pp. 378–387.
[8]Grimble, M. J., "Design of Optimal Output Regulators Using Multivariable Root-loci," *IEE Proceedings*, Vol. 128, Pt. D, March 1981, pp. 41–49.
[9]Pal, J. K. and Mahalanabis, A. K., "Optimal Stationary Feedback Control with Specified Relative Stability," *Proceedings of the IEE*, Vol. 120, April 1973, pp. 509–513.
[10]Letov, A. M., "Analytical Controller Design, I, II," *Automatic Remote Control*, Vol. 21, April 1960, pp. 303–306.
[11]Lewis, F. L., *Optimal Control*, Wiley-Interscience, New York, 1986.
[12]Misra, P. and Patel, R. V., "Computation of Transfer Function Matrices of Linear Multivariable Systems," *Automatica*, Vol. 23, 1987, pp. 635–640.
[13]Kailath, T., *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, Sept. 1980.
[14]Amin, M. H., "Optimal Discrete Systems with Prescribed Eigenvalues," *International Journal of Control*, Vol. 40, Oct. 1984, pp. 783–794.

# Problem of the Dynamics of a Cantilever Beam Attached to a Moving Base

S. Hanagud* and S. Sarkar†
*Georgia Institute of Technology, Atlanta, Georgia*

## Introduction

In a recent article in this journal, Kane et al.[1] formulated a comprehensive theory of a cantilever beam mounted on a moving support. They took into account the stretch, bending in two principal directions, shear deformations, and warping of the beam. The formulation is based on a method that has been attributed to Kane. Kane's method,[2] which is closely related to Gibbs' method, provides a systematic procedure and

Received Jan. 4, 1988; revision received March 22, 1988. 

*Professor, School of Aerospace Engineering.
†Graduate Research Assistant, School of Aerospace Engineering.